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5

TERMINOLOGY

addition principle
combination
factorial
inclusion-exclusion principle
multiplication principle
Pascal's triangle
permutation

COMBINATORICS

COUNTING METHODS AND COMBINATIONS

Prior learning

THE INCLUSION-EXCLUSION PRINCIPLE FOR THE UNION OF TWO SETS AND THREE SETS

- determine and use the formulas for finding the number of elements in the union of two and the union of three sets. (ACMSM005)

COMBINATIONS (UNORDERED SELECTIONS)

- solve problems involving combinations (ACMSM007)
- use the notation $\binom{n}{r}$ or ${}^nC_r$ (ACMSM008)
- derive and use simple identities associated with Pascal's triangle. (ACMSM009)

5.01 SIMPLE COMBINATIONS

You studied permutations in Chapter 3. You can think of a permutation as an *ordered* list. Now imagine a child's box of different coloured pencils. To colour the sky in a picture, it doesn't matter where the blue pencil is in the box. All that matters is that there *is* a blue one in the set of pencils. So the order of the colours in a list of the pencils is irrelevant. A list like this, where the order is unimportant, is called a **combination**. You may have already done some work with combinations in Maths Methods.

IMPORTANT

A **combination** is unordered.

A combination of a set of symbols is a selection of none, some or all of the symbols.

The number of combinations of r objects from n distinct objects is given by

$$\begin{aligned} {}^nC_r = \binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \end{aligned}$$

Proof

If you have a combination of r objects, then there are $r!$ permutations of them.

Thus ${}^nP_r = {}^nC_r \times r!$ so ${}^nC_r = \frac{{}^nP_r}{r!}$, and substituting ${}^nP_r = \frac{n!}{(n-r)!}$, you get the formula. **QED**

There are usually far fewer combinations than permutations. For example, if you picked the top 3 students from 5 to enter a competition, there would be 10 possible combinations of 3 students. However, if the order mattered, there would be 60 possible permutations.

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Example 1

Calculate the value of ${}^{8}C_{3}$.

Solution

Write the formula.

$${}^{n}C_{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Find the last term of the numerator.

$$n - r + 1 = 8 - 3 + 1 = 6$$

Substitute values.

$${}^{8}C_{3} = \frac{8\times7\times6}{1\times2\times3}$$

Cancel where possible and evaluate.

$$= \frac{8\times7\times\cancel{6}^{\cancel{3}\,1}}{1\times{}_{1}\cancel{2}\times{}_{1}\cancel{3}} = 56$$

Write the answer.

$${}^{8}C_{3} = 56$$

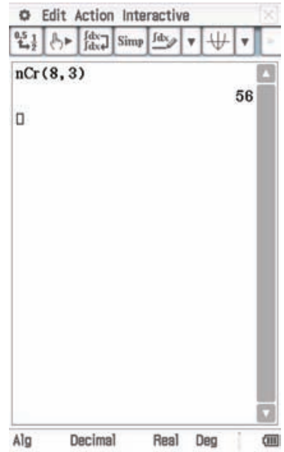
TI-Nspire CAS

Use a Calculator page.
Type nCr() or use menu, 5: Probability and 3: Combinations. Insert n = 8 and r = 3 separated by a comma and press enter.

ClassPad

Use the Main $\sqrt{\alpha}$ application.
Press Keyboard and tap ▼.
Tap ▶ as necessary to scroll along and tap **N**.
Tap **nCr(** .

Enter the values of n, then r, separated by a comma.
Close the brackets and tap EXE or press EXE.



Example 2

How many different committees of four people can be chosen from a group of 15?

Solution

Write the formula. $${}^{n}C_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Substitute the values in. $${}^{15}C_4 = \frac{15\times14\times13\times12}{1\times2\times3\times4} = 1365$$

Write the answer. There are 1365 different committees.

Example 3

Combination calculations

Find the value of n if ${}^{n+1}C_6 = {}^{n}C_5$.

Solution

Write the formula. $$\frac{(n+1)!}{6!(n+1-6)!} = \frac{n!}{5!(n-5)!}$$

Expand the formula. $$\frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\times2\times3\times4\times5\times6} = \frac{n(n-1)(n-2)(n-3)(n-4)}{1\times2\times3\times4\times5}$$

Cancel where possible. $$\frac{(n+1)}{6} = \frac{1}{1}$$

Solve for n. $n = 5$

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EXERCISE 5.01 Simple combinations

Concepts and techniques

Combinations

1 Example 1 Calculate the values in **a**, **b** and **c** without using your calculator. Simplify the expressions in **d** and **e**.

a $^{6}C_2$ b $^{8}C_8$ c $^{25}C_1$ d $^{n}C_3$ e $^{n+1}C_2$

2 CAS Work out the values of the following.

a $^{8}C_3$ b $^{10}C_6$ c $^{12}C_4$ d $^{13}C_5$ e $^{13}C_8$

3 Example 2 Find the number of different ways that a committee of 6 people can be made randomly from a group of:

a 8 people b 9 people c 11 people
d 15 people e 20 people.

4 Example 3 Find n if:

a $^{n}C_2 = 6$ b $^{n+1}C_3 = {}^{n}C_2$ c $^{n}C_2 = 45$
d $^{n+1}C_7 = {}^{n}C_6$ e $^{n}C_2 = 66$ f $^{n}C_5 = {}^{n}C_{11}$

Reasoning and communication

5 How many different selections of five cards can be made from a deck of 52 different cards?

6 Ten people are introduced and shake hands with each other.

a How many handshakes are there?
b How many handshakes are there for n people?

7 How many triangles can be drawn using five given points on the circumference of a circle?

8 A short test has ten questions, all of which must be answered Yes or No. In how many sets of answers can a person get half the questions wrong?

9 A coin is tossed 20 times. How many different arrangements are there for tossing 5 heads?

10 A set of 10 different coloured marbles are placed in a bag and 6 are selected at random. In how many different ways can this happen?

11 A bag contains 12 different lollies with blue wrappers and 15 different lollies with red wrappers. If I take a handful of 6 lollies out of the bag, how many different handfuls are possible?

12 A car has a radio with five push-buttons that can be tuned to different stations. In a city that has 12 radio stations, how many different selections of those stations can be chosen for the push-buttons if they are placed in order of the wavelength?

13 A simple Lotto game uses the numbers 1–40, with six numbers being chosen. How many different ways can the numbers be chosen?

14 Show that $\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$

15 Show that

a $\binom{13}{7} = \binom{13}{6}$ b $\binom{n}{r} = \binom{n}{n-r}$

5.02 USING COMBINATIONS

Sometimes you have to reduce the number of combinations to allow for restrictions.

Example 4

A committee of five people is to be chosen at random from a class of 20.

a How many different committees could be chosen?

b How many different committees that include Johanna and Esrif could be chosen?

Solution

a Use the formula.

$$\text{Number of committees} = {}^{20}C_5 = 15\,540$$

b Committees with both will have 3 people from the other 18.

$$\text{Number including Esrif and Johanna} = {}^{18}C_3 = 816$$

You will often need to use the multiplication principle with combinations to solve a problem. You should remember this from your earlier work in Chapter 3.

IMPORTANT

The **multiplication principle** says that for a choice made in two stages with a ways for one part and b ways for the second part, there are $a \times b$ choices altogether.

For a choice made in n stages: if there are a_1 ways for the first part, a_2 ways for the second part, a_3 ways for the third part, and so on, then there are $a_1 a_2 a_3 \ldots a_n$ choices altogether.

Example 5

A team of 6 men and 5 women is chosen at random from 10 men and 9 women. How many different possible teams include Kaye, Stella and Peter?

Solution

The male combinations including Peter have 5 other men.

Male combinations with Peter = ${}^{9}C_5$

Female combinations including Kaye and Stella have 3 others.

Female combinations with Kaye and Stella = ${}^{7}C_3$

Use the multiplication rule.

Number of teams with all three = ${}^{9}C_5 \times {}^{7}C_3$

Use your calculator.

$= 126 \times 35$
$= 4410$

Write the answer.

There are 4410 possible teams that include Kaye, Stella and Peter.

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You may need to use the addition principle with combinations to solve a problem.

IMPORTANT

The **addition principle** says that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.

Remember that mutually exclusive sets have no intersection; they have no elements in common with one another.

Example 6

The executive committee, comprising the president, secretary, treasurer and three other members of a club are elected at the annual general meeting, which is attended by a total of 22 people, including three members of the former executive committee. How many possible committees could be elected from those attending that include at least one of the former executive?

Solution

State what has to be found.	The number of combinations that include 1, 2 or 3 of the former executive committee members is needed.
How many others have to be elected with one of the former executive committee members?	To include exactly one particular member of the former executive committee members, 5 out of the other 19 people need to be elected. Number including a particular one $= {}^{19}C_5$ $= 11\,628$
There are 3 possible former members.	Number of combinations including one only $= 3 \times 11\,628$ $= 34\,884$
How many combinations are there with two of the former executive committee members?	To include exactly 2 particular members of the former executive, 4 more are to be elected. Number with two $= {}^{19}C_4 = 3876$
There are 3 ways to choose 2 of the former 3.	Number of combinations including two $= 3 \times 3876$ $= 11\,628$
There is only one way to include all three.	Number of combinations including three $= {}^{19}C_3$ $= 969$
Use the addition principle.	Number of possible committees including at least one of the former executive committee $= 34\,884 + 11\,628 + 969$ $= 47\,481$

INVESTIGATION Poker hands

In the card game of poker, players try to get special combinations of 5 cards. The highest possible combination is Ace-King-Queen-Jack-10 of the same suit. This is called a royal flush.

How many possible poker hands are there? How many are royal flushes? What is the probability of getting a royal flush?

The other combinations are as follows.

- Straight flush: 5 cards in a row of the same suit such as 3-4-5-6-7 of spades
- Four of a kind: 4 cards all the same such as 10-10-10-10-Q
- Full house: 3 cards the same and 2 others the same such as 7-7-7-A-A
- Flush: 5 cards of the same suit such as 3, 6, 8, J, Q of clubs
- Straight: 5 cards in a row, such as 7-8-9-10-J
- Three of a kind: 3 cards the same, such as J-J-J-A-3
- Two pair: Two lots of 2 cards the same such as Q-Q-6-6-9
- A pair: 2 cards the same, such as K-K-Q-7-6

Work in groups to determine the number and probabilities of each combination. Compare the probabilities to the order of the hands from highest to lowest. What do you find?

EXERCISE 5.02 Using combinations

Reasoning and communication

1 Example 4 An amateur theatrical company is forming a new committee for the following year. Since nobody really wants to do the committee work, it is decided to select the five-member committee at random from the 20 full members of the company. It is generally agreed that no committee should include both Jodie and Michael, as they do not work well together.
   a How many possible committees would there be, regardless of who is on the committee?
   b How many possible committees would include both of them?

2 Example 5 A company board has ten directors. Six of the directors are in favour of a merger proposal while four are opposed to it. A subcommittee of three directors is to be chosen to investigate the merger proposal. How many subcommittees can be selected so that exactly two directors are in favour of the proposal?

3 Example 6 A family has 19 movies on DVD, including all three of the 'Back to the Future' movies. How many selections of 5 movies would include at least one of the 'Back to the Future' DVDs?

4 The school team has 10 players available from the netball squad for the next game. A netball team has 7 players.
   a How many possible teams are there?
   b How many of the possible teams would include Sonja or Renee or both?

5 Of the players in the squad for the premier school cricket team, 5 are primarily fast bowlers, 2 are primarily spin bowlers, 7 are primarily batsmen and there is one all-rounder. How many different teams of 11 could be chosen that included the all-rounder and at least one spin bowler?

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6 A pet shop owner wishes to display five animals in the shop window. There are six cats and three dogs available for display. How many combinations of five display animals can be chosen so as to include:

a one dog?

b at least one dog?

Alamy/John Norman

7 A committee of 6 people is to be selected randomly from a group of 11 men and 12 women. Find the number of possible committees if:

a there is no restriction on who is on the committee

b all committee members are to be male

c all members are to be female

d there are to be 3 men and 3 women

e a particular woman is included

f a particular man is not included

g there are to be 4 women and 2 men.

8 A horserace has 15 competing horses. At the TAB, a quinella pays out on the horses that come in first and second, in either order. Ryan decides to bet on all possible combinations of quinellas. If it costs him $1 a bet, how much does he pay?

9 A group of 25 students consists of 11 who play a musical instrument and 14 who don't. Find the number of different arrangements possible if a group of 9 students is selected at random:

a with no restriction

b who all play musical instruments

c where 5 play musical instruments

d where 2 don't play musical instruments.

10 A set of cards consists of 8 yellow and 7 red cards.

a If 10 cards are selected at random, find the number of different combinations possible.

b If 8 cards are selected, find the number of combinations of:

i 4 yellow cards ii 6 yellow cards iii 7 yellow cards iv 5 red cards.

11 Ten cards are randomly selected from a set of 52 playing cards. Find the number of combinations selected if:

a there are no restrictions (answer in scientific notation, correct to 3 significant figures)

b they are all hearts

c there are 7 hearts

d they are all red cards

e there are 4 aces.

12 An animal refuge has 17 dogs and 21 cats. If a nursing home orders 12 animals at random, find the number of ways that the order would have

a 7 dogs b 9 dogs c 10 dogs d 4 cats e 6 cats.

13 There are 8 white, 9 red and 5 blue marbles in a bag and 7 are drawn out at random. Find the number of selections possible:

a with no restriction

b if all marbles are red

c if there are 3 white and 2 red marbles

d if there are 4 red and 1 blue marble

e if there are 4 white and 2 blue marbles.

14 Out of a group of 25 students, 7 walk to school, 12 catch a train and 6 catch a bus. If 6 students are selected, find the number of combinations if:

a all walk to school

b no one catches a bus

c 3 walk to school and 1 catches a bus

d 1 walks to school and 4 catch a train

e 3 catch a train and 1 catches a bus.

15 At a karaoke night, a group of 14 friends decide that 4 of them will sing a song together. Of the friends, 5 have previously sung this song before. In how many ways can they do this if they select:

a friends who have all sung the song previously

b 2 of the friends who sang the song previously

c none of the friends who sang the song previously?

16 A cricket team of 11 players is to be chosen from a squad of 15 players. How many different teams can be chosen that include the following?

a Sam and David

b Sam, David and Paresh

c Sam, David, Paresh and Akmal

d Sam or David

5.03 PASCAL'S TRIANGLE

Pascal's triangle is named after Blaise Pascal (1623–1662), although it is also called Yang Hui's triangle in China, Tartaglia's triangle in Italy and Khayyam's triangle in Iran after earlier mathematicians who studied its properties.

The first 7 rows are shown on the right. The rows are numbered from 0 down.

The numbers are staggered in each successive row. You can work them out by adding the numbers diagonally above each position, as shown by the arrows. If there is no number above, it is taken as 0. Work out the next few rows of the triangle for yourself.

1	Row 0
1 1	Row 1
1 2 1	Row 2
1 3 3 1	Row 3
1 4 6 4 1	Row 4
1 5 10 10 5 1	Row 5
1 6 15 20 15 6 1	Row 6
	Row 7?
	Row 8?
	Row 9?

The numbers in a row are counted across from 0, so element number 3 in row 6 is 20. However the third element in row 6 is 15. This can be quite confusing, so you always need to be clear whether you are referring to a row or element by its number or its position.

The symbol Pa is sometimes used to refer to numbers in Pascal's triangle. Using this notation, $Pa_{4,2} = 6$ is the centre element in the row that begins 1, 4, …

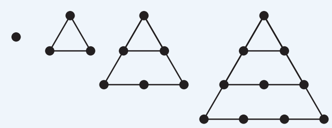
INVESTIGATION Patterns in Pascal's triangle

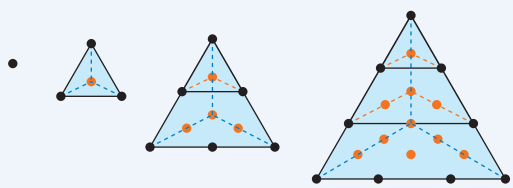
Look at the diagonals of Pascal's triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Notice that the first two diagonals are just one and the counting numbers.

Compare the numbers in the diagonals with the number of dots in each of the following sequences of shapes.





Is there a shape for the next diagonal? Can you draw it?

Now look at the digits in the rows as numbers: 1, 11, 121, 1331, …

How is 121 related to 11?

How is 1331 related to 121?

Does the pattern continue?

Can you make it so for rows with double digit numbers? How?

The numbers in Pascal's triangle are actually the same as the numbers for combinations.

IMPORTANT

The numbers in Pascal's triangle are the combinatorial numbers given by ^nC_r.

For example, counting from 0, the third number in row 6 is 20.

It is also the case that $^6C_3 = 20$.

Apart from the ends, any number in any row of Pascal's triangle is the sum of the two diagonally above. Looking at rows 4 and 5 of Pascal's triangle, the first 10 in row 5 is the sum of the 4 and 6 above it.

$$\begin{array}{ccccccccccc} & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

It is true that ${}^4C_1 = 4$, ${}^4C_2 = 6$, ${}^5C_2 = 10$, and ${}^4C_1 + {}^4C_2 = {}^5C_2$ $(4 + 6 = 10)$, but is this always going to work for the combinations? For that to be true, it would have to be true that ${}^{n-1}C_{m-1} + {}^{n-1}C_m = {}^nC_m$, no matter what the values of n and m were (with $m < n$).

In the case above, $n = 5$ and $m = 2$.

What about $n = 8$ and $m = 6$? Does ${}^7C_5 + {}^7C_6 = {}^8C_6$, corresponding to the third last number on row of Pascal's triangle?

${}^7C_5 = 21$, ${}^7C_6 = 7$ and ${}^8C_6 = 28$, so it does work in this case as well.

The proof that the numbers in Pascal's triangle are the combinatorial numbers, ${}^nC_r$ is shown below.

Proof

${}^0C_0 = \dfrac{0!}{0! \times 0!} = 1$, the very top row of Pascal's triangle.

Consider the other rows. For each row:

${}^nC_0 = 1$, the number on the left of row n.

${}^nC_n = 1$, the number of the right of row n of Pascal's triangle.

Thus the numbers at the ends of each row are correct.

Now consider a number somewhere in the middle.

$$\begin{aligned}
{}^{n-1}C_{m-1} + {}^{n-1}C_m &= \frac{(n-1)!}{(m-1)![(n-1)-(m-1)]!} + \frac{(n-1)!}{m![(n-1)-m]!} \\
&= \frac{(n-1)!}{(m-1)![n-1-m+1]!} + \frac{(n-1)!}{m![n-1-m]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m)!} + \frac{(n-1)!}{m![n-m-1]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m)(n-m-1)!} + \frac{(n-1)!}{m(m-1)![n-m-1]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!}\left[\frac{1}{(n-m)} + \frac{1}{m}\right] \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!}\left[\frac{m+n-m}{(n-m)\times m}\right] \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!} \times \frac{n}{m \times (n-m)} \\
&= \frac{n!}{m!(n-m)!} = {}^nC_m
\end{aligned}$$

Therefore ${}^{n-1}C_{m-1} + {}^{n-1}C_m = {}^nC_m$

Since it works for the first rows, this shows that combinations follow the same rule as for the numbers in Pascal's triangle, so they must be the same numbers. **QED**

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Example 7

Prove that the second number in row n of Pascal's triangle is always n for $n > 0$.

Solution

Write the equivalent combinatorial formula, remembering that the second number is $\mathrm{Pa}_{n,\,1}$.	**Proof** $\mathrm{Pa}_{n,\,1} = {}^{n}C_1$
Substitute in the formula.	$= \dfrac{n!}{1!(n-1)!}$
Write $n! = n(n-1)!$	$= \dfrac{n(n-1)!}{1\times(n-1)!}$
Simplify.	$= n$ **QED**

In Maths Methods, you may have seen the expansion

$$(x+y)^5 = {}^5C_5x^5 + {}^5C_4x^4y + {}^5C_3x^3y^2 + {}^5C_2x^2y^3 + {}^5C_1xy^4 + {}^5C_0y^5$$
$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

In Maths Methods, you were *given* the fact that the coefficients of $(x+y)^n$ are the values ${}^nC_r$.

However, this was not properly proved.

IMPORTANT

The binomial coefficients, the coefficients in the expansion of $(x+y)^n$, are given by ${}^nC_r$.

Proof

Consider the x^2y^3 products in the expansion of

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

Each x^2y^3 product is obtained by multiplying the x terms from 2 of the 5 brackets and y terms from the rest of the brackets, so you are choosing 2 xs from 5 possible xs and there are ${}^5C_2$ ways to do this.

Now consider the general case.

Consider x^ry^{n-r} products in the expansion of

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

Each product is obtained by multiplying x terms from r of the brackets and y terms from the rest of the brackets.

The number of products is precisely the number of ways of choosing r of the brackets to get the x terms. This is ${}^nC_r$.

Hence $(x+y)^n = {}^nC_0x^n + {}^nC_1xy^{n-1} + {}^nC_2x^2y^{n-2} + \cdots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$ **QED**

A similar method can be applied to other expansions.

Example 8

Use combinations to expand $(x+y+z)^4$.

Solution

Write out the problem. $(x+y+z)^4$

Write out the power. $=(x+y+z)(x+y+z)(x+y+z)(x+y+z)$

Write the possible terms.

$$\begin{aligned}&= a_1x^4 + a_2x^3y + a_3x^3z + a_4x^2y^2 + a_5x^2yz + a_6x^2z^2\\ &\quad + a_7xy^3 + a_8xy^2z + a_9xyz^2 + a_{10}xz^3 + a_{11}y^4\\ &\quad + a_{12}y^3z + a_{13}y^2z^2 + a_{14}yz^3 + a_{15}z^4\end{aligned}$$

where $a_1, a_2, \ldots$ are constants.

Write the constants.

$a_1 = {}^4C_4$, the number of combinations of 4 xs (leaving 0 ys and 0 zs) from 4 brackets.

Similarly $a_{11} = a_{15} = {}^4C_4 = {}^4C_0$, the number of combinations of 4 ys or 4 zs from 4 brackets.

$a_2 = {}^4C_3$, the number of combinations of 3 xs (leaving 1 y) from 4 brackets.

Similarly $a_3 = a_7 = a_{10} = a_{12} = a_{14} = {}^4C_1 = {}^4C_3$ the number of combinations of 3 of one symbol (leaving 1 of another) from the 4 brackets.

$a_4 = a_6 = a_{13} = {}^4C_2$, the number of combinations of 2 of one symbol (leaving 2 of another) from 4 brackets.

Finally, $a_5 = a_8 = a_9 = {}^4C_2 \times {}^2C_1$, the number of combinations of 2 of one symbol from 4 brackets and 1 of another symbol from the remaining 2 brackets (leaving 1 of the third symbol).

Calculate the values.

$a_1 = a_{11} = a_{15} = {}^4C_0 = {}^4C_4 = 1$

$a_2 = a_3 = a_7 = a_{10} = a_{12} = a_{14} = {}^4C_1 = {}^4C_3 = 4$

$a_4 = a_6 = a_{13} = {}^4C_2 = 6$

$a_5 = a_8 = a_9 = {}^4C_2 \times {}^2C_1 = 6 \times 2 = 12$

Write the answer.

$$\begin{aligned}(x+y+z)^4 &= x^4 + 4x^3y + 4x^3z + 6x^2y^2 + 12x^2yz + 6x^2z^2\\ &\quad + 4xy^3 + 12xy^2z + 12xyz^2 + 4xz^3\\ &\quad + y^4 + 4y^3z + 6y^2z^2 + 4yz^3 + z^4\end{aligned}$$

The combination numbers, which are the numbers in Pascal's triangle, have many other interesting properties.

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EXERCISE 5.03 Pascal's triangle

Reasoning and communication

1 Example 7 Show that the third number of any row of Pascal's triangle is always $\frac{1}{2}n(n-1)$ for $n > 1$.

2 The Fibonacci numbers are obtained by starting with the numbers 1 and 1 and adding every two numbers to get the next one. Thus they are 1, 1, 2, 3, 5, … as $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on. Show that the 'shallow diagonals' of Pascal's triangle sum to the Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

3 Show that the numbers in any row of Pascal's triangle are symmetrical.

4 Show that for any row the rth number is $\frac{n-r}{r+1}$ times the $(r-1)$th number, that is, ${}^nC_{r+1} = \frac{n-r}{r+1} \times {}^nC_r$, assuming that ${}^nC_0 = 1$ and $n > 0$.

5 Show that the sum of row n of Pascal's triangle is 2^n. (Hint: use the binomial expansion).

6 Example 8 Use combinations to expand $(x + y + z)^3$.

7 When the numbers in the diagonal part (1, 4, 10, 20) are added, they give 35, which is on the next row in the opposite diagonal direction. Show that this is always true for a hockey stick pattern like this.

1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

8 Show that the products of the numbers in each triangle in the 'Star of David' patterns as shown below are the same.

1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

In the examples, $6 \times 5 \times 20 = 4 \times 15 \times 10$ and $21 \times 70 \times 84 = 35 \times 126 \times 28$.

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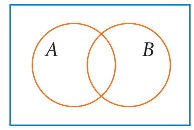
5.04 THE INCLUSION-EXCLUSION PRINCIPLE

How can you count arrangements that can be divided into different parts that are *not* mutually exclusive? You need to allow for arrangements that are counted twice.

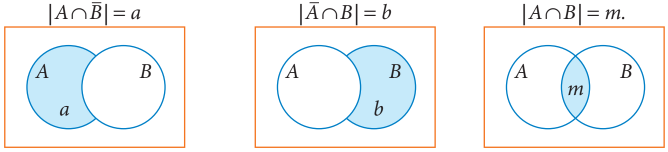


Consider the number of elements in $A \cup B$, that is $|A \cup B|$ or $n(A \cup B)$.

The Venn diagram shows there could be elements in both A and B.

Suppose that there are a elements in A that are not in B, b elements in B that are not in A, and m elements in $A \cap B$. This is shown below.

$|A \cap \bar{B}| = a$ $\quad |\bar{A} \cap B| = b$ $\quad |A \cap B| = m.$



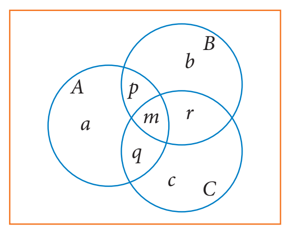
It is clear that $|A \cup B| = a + b + m = (a + m) + (b + m) - m = |A| + |B| - |A \cap B|$.

This is the simplest case of the **inclusion-exclusion principle**.

The number of elements in the union of two sets *includes* the number in each set, and *excludes* the number counted twice because they are in both sets (the intersection of the two).

For example, if there are 12 blue-eyed people and 10 people with fair hair in a class, and 5 of those with blue eyes have fair hair, then there must be $12 + 10 - 5 = 17$ people with either blue eyes, fair hair or both.

What about three sets?



Suppose there are a elements in A that are not in B or C ,

b elements in B that are not in A or C,

c elements in C that are not in A or B,

p elements in $A \cap B$ that are not in C,

q elements in $A \cap C$ that are not in B,

r elements in $B \cap C$ that are not in A,

and m elements in $A \cap B \cap C$.

Clearly

$|A \cup B \cup C|$

$$= a + b + c + p + q + r + m$$

$$= (a + p + m + q) + (b + p + m + r) + (c + q + m + r) - (p + m) - (q + m) - (r + m) + m$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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This is the inclusion-exclusion principle for three sets.

The number of elements in the union of three sets *includes* the number in each set, and twice *excludes* the number counted three times because they are in the intersections of pairs of sets and *includes* the number added and subtracted three times because they are in all three sets.

If 10 students take geography, 9 study history, 7 do art, 6 take geography and history, 5 study geography and art, 4 do history and art and 2 take all three, then there must be $10 + 9 + 7 - 6 - 5 - 4 + 2 = 13$ that study geography, art, history or a combination of some of these subjects. You should satisfy yourself that this example of the principle is true using a Venn diagram.

IMPORTANT

Inclusion-exclusion principle

The number of elements in a union of sets is:

- for 2 sets: $|A \cup B| = |A| + |B| - |A \cap B|$
- for 3 sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The pattern of inclusions and exclusions is clear and extends to any number of sets.

While the pattern for the inclusion-exclusion principle can be extended to any number of sets, you are only required to use it for unions involving 2 or 3 sets.

Example 9

How many counting numbers up to 100 are divisible by 5 or 6?

Solution

Count the numbers divisible by 5.	No. divisible by $5 = 100 \div 5 = 20$
Count the numbers divisible by 6.	No. divisible by $6 = 100 \div 3 = 16\frac{2}{3}$, so 16
Count those divisible by 5 and 6.	No. divisible by $30 = 100 \div 30 = 3\frac{1}{3}$, so 3
Use inclusion-exclusion.	No. divisible by 5 or $6 = 20 + 16 - 3$
Calculate the result.	$= 33$
Write the answer.	33 counting numbers up to 100 are divisible by 5 or 6.

In Example **9**, if the question had asked for 'numbers under 100 divisible by 5 or 6', then 100 itself, which is divisible by 5, would be excluded so the number would be 32.

Example 10

How many positive whole numbers under 1000 are divisible by 2, 3 or 5?

Solution

Count the numbers divisible by 2.	No. divisible by $2 = 999 \div 2$ $= 499.5$, so 499
Count the numbers divisible by 3.	No. divisible by $3 = 999 \div 3 = 333$
Count the numbers divisible by 5.	No. divisible by $5 = 999 \div 5$ $= 199.8$, so 199
Count those divisible by 2 and 3.	No. divisible by $6 = 999 \div 6$ $= 166\frac{1}{2}$, so 166
Count those divisible by 2 and 5.	No. divisible by $10 = 999 \div 10 = 99.9$, so 99
Count those divisible by 3 and 5.	No. divisible by $15 = 999 \div 15$ $= 66.6$, so 66
Count those divisible by 2, 3 and 5.	No. divisible by $30 = 999 \div 30$ $= 33.3$, so 33
Use inclusion-exclusion.	No. divisible by 2, 3 or 5 is $499 + 333 + 199 - 166 - 99 - 66 + 33$
Calculate the result.	$= 733$

EXERCISE 5.04 The inclusion-exclusion principle

WS The inclusion-exclusion principle

Concepts and techniques

1 Example 9 How many counting numbers up to and including 100 are multiples of 2 or 3?

2 How many natural numbers under 2000 are multiples of 5 or 7?

3 How many natural numbers up to and including 1500 are multiples of 3 or 5?

4 Example 10 How many whole positive numbers under 1000 are multiples of 2, 3 or 7?

5 How many three-digit counting numbers (numbers from 100 to 999) are multiples of 2 or 3?

6 How many three-digit numbers are multiples of 2, 3 or 4?

7 How many three-digit numbers are multiples of 2, 3 or 7?

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8 How many four-digit numbers are multiples of 2, 3 or 5?

9 How many numbers from 300 to 700 inclusive are multiples of 3 or 5?

10 How many numbers from 200 to 800 inclusive are multiples of 2, 3 or 5?

11 How many numbers from 600 to 2400 inclusive are multiples of 3, 5 or 7?

Reasoning and communication

12 17 of the houses in a short street have both fluorescent tubes and fluorescent bulbs. 10 have fluorescent tubes, fluorescent bulbs and some incandescent bulbs. 21 have fluorescent tubes and incandescent bulbs and 19 have incandescent and fluorescent bulbs. 30 of the houses have incandescent bulbs, 30 have fluorescent tubes and 30 have fluorescent bulbs. All the houses have fluorescent or incandescent lights. How many houses are in the street?

5.05 SIMPLE APPLICATIONS TO PROBABILITY

The inclusion-exclusion principle may be used in combination with other counting methods.

Example 11

The first buses going to the school swimming carnival are carrying competitors. The first bus has 45 students entered in the 50 m events and 35 entered in other events. 28 students are entered in both the 50 m and other events. The fourth (last) teacher got on the bus, counted the number on board, and said ‘That’s it, no more legally allowed on this bus!”

a What is the legal carrying capacity of the bus?

b What is the probability that a randomly selected student is entered in a 50 m event?

Solution

a Use the inclusion-exclusion principle.

$$\begin{aligned}\text{Number of students} &= 45 + 35 - 28\\ &= 52\end{aligned}$$

Add the teachers.

$$\begin{aligned}\text{Legal carrying capacity} &= 52 + 4\\ &= 56\end{aligned}$$

b Use the rule for probability.

$$\begin{aligned}P(50\text{ m event}) &= \frac{45}{52}\\ &\approx 0.8654\end{aligned}$$

Example 12

Out of 50 people, 39 like vanilla ice-cream, 33 like strawberry ice-cream and 32 like chocolate ice-cream. 27 like vanilla and strawberry, 23 like vanilla and chocolate and 22 like chocolate and strawberry. What is the probability that someone chosen at random will like all three flavours, given they all like at least one flavour?

Solution

Choose letters for the sets of people who like each flavour.	Let V = people who like vanilla S = people who like strawberry and C = people who like chocolate
Write the inclusion-exclusion principle.	$\lvert V \cup S \cup C\rvert = \lvert V\rvert + \lvert S\rvert + \lvert C\rvert - \lvert V \cap S\rvert - \lvert V \cap C\rvert - \lvert S \cap C\rvert + \lvert V \cap S \cap C\rvert$
Substitute in the values.	$50 = 39 + 33 + 32 - 27 - 23 - 22 + \lvert V \cap S \cap C\rvert$
Simplify.	$50 = 32 + \lvert V \cap S \cap C\rvert$
Solve.	$\lvert V \cap S \cap C\rvert = 18$
Find the probability.	$P(\text{all flavours}) = \frac{18}{50} = 0.36$
Write the answer.	The probability that the person will like all three flavours is 0.36.

EXERCISE 5.05 Simple applications to probability

Reasoning and communication

1 Example 11 A 'project' home builder offers the following interior floor coverings in the living areas of its standard designs: carpets from a range of 8 different patterns, or tiles from a range of 6. In homes built in the last 6 months, 28 clients chose carpet, 32 chose to use tiles and 16 chose carpet in some rooms and tiles in others.
   - a How many houses were built?
   - b What is the probability of randomly choosing one of the houses with both carpet and tiles?

2 Example 12 A buffet supper offers pasta salad, rice salad and garden salad with a choice of meats. Sixty of the people present took at least some salad, with 5 taking some of all three salads. 28 people took rice salad and 32 people took pasta salad. 15 took both garden and rice salad, 10 took both pasta and rice salad and 20 took both pasta and garden salad. What is the probability that a person chosen at random from those that took salad had some garden salad?

3 A child likes to play the 'guess my number' game with her older relatives. One is a Year 11 maths student, who tries to improve his chances by saying that she is only allowed to pick numbers from 1 to 100 inclusive that do not divide by 2, 3, 4 or 5. What is the probability of randomly guessing one of these numbers?

4 Three people went into the art gallery together on a rainy day and all three had black umbrellas. Because it was a major exhibition, and because of the rain, there were lots of umbrellas, coats, hats and bags in the cloakroom. The cloakroom attendant assumed that since they came in

together, they were together, and put all three umbrellas together in the last 'box' space in the cloakroom. When they left, they were given one each at random.

a What is the probability that they all got the correct umbrella?

b What is the probability that they all got the wrong umbrella?

5 A waiter on his first day became nervous and could not remember who ordered what at a table of four people who had all ordered different dishes. Instead of asking, he tried to bluff his way out by putting the meals down at random.

a What is the probability that he got them all right?

b What is the probability that he got them all wrong?

6 An assistant in an antiques store was dusting and accidentally knocked the price cards off four possum (colonial) dressers. The dressers were priced at \$1200, \$1300, \$2700 and \$2800. Not wanting to make a fuss, the assistant put the cards back at random. The assistant then sold one of the dressers (now priced at \$1200) to a customer before the owner came back from lunch.

a What is the probability that all the prices went back correctly?

b What is the probability that \$1200 was the right price?

c What is the probability that all the prices were wrong?

d What is the probability that the customer got a huge bargain?

5.06 GENERAL USE OF COUNTING METHODS

You have now studied various counting methods and related topics: the multiplication principle, the addition principle, the pigeonhole principle, permutations with and without repetition, combinations, Pascal's triangle and the inclusion-exclusion principle.

You need to be able to choose the most appropriate methods to solve problems and may have to use several methods together.

You should remember the following rules for permutations.

IMPORTANT

The number of ways of arranging n **objects in an ordered list** is given by

$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

The number of permutations of r objects from n different objects is written as ${}^nP_r$, which is calculated using the formula

$${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).$$

The number of **circular permutations** of n different objects or symbols is $(n-1)!$.

The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$.

If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \cdots \leq n$.

Example 13

How many different 4-letter 'words' can be made from the letters of ALMANAC if, in making a word, each of the seven letters can be used only once in each 'word'?

Solution

Count the combinations of four different letters.	Number of combinations of different letters $= {}^5C_4 = 5$
Count their permutations.	Permutations of four different symbols $= 4! = 24$
Use the multiplication principle.	Number of words with four different letters $= 5 \times 24$
Count the combinations with two As.	Number of combinations with two As $= {}^4C_2 = 6$
Count their permutations.	Permutations of 4 symbols with 2 the same $= \dfrac{4!}{2!} = 12$
Use the multiplication principle.	Number of words with two As $= 6 \times 12$
Count the combinations with three As	Number of combinations with three As $= 4$
Count their permutations.	Permutations of 4 symbols with 3 the same $= \dfrac{4!}{3!} = 4$
Use the multiplication principle.	Number of words with three As $= 4 \times 4$
Use the addition principle.	Total number of words $= 5 \times 24 + 6 \times 12 + 4 \times 4$
Calculate the answer.	$= 208$
State the answer.	208 different 'words' can be made from the letters of ALMANAC using each letter only once.

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Example 14

How many different 5-digit whole numbers can be made from the digits 2, 3, 4, 4, 4, 5, 7, 7, 9 if each of the 9 digits can be used only once in each number?

Solution

Count the combinations with no repeats.

Combinations with no repeats $= {}^6C_5 = 6$

Count their permutations.

Permutations of 5 symbols $= 5! = 120$

Count the numbers with no repeated digits.

Number with no repeats $= 6 \times 120 = 720$

Count the combinations with two 4s.

Combinations with two 4s $= {}^5C_3 = 10$

Count their permutations.

Permutations of 5 symbols with 2 the same $= \dfrac{5!}{2!} = 60$

Count the numbers with two 4s.

Number with two 4s $= 10 \times 60 = 600$

It is the same for the two 7s.

Number with two 7s $= 10 \times 60 = 600$

Count the combinations with three 4s.

Combinations with three 4s $= {}^5C_2 = 10$

Count their permutations.

Permutations of 5 symbols with 3 the same $= \dfrac{5!}{3!} = 20$

Count the numbers with three 4s.

Number with three 4s $= 10 \times 20 = 200$

Count the combinations with two 4s and two 7s.

There are 4 combinations with two 4s and two 7s

Count their permutations.

Permutations of two 4s and two 7s $= \dfrac{5!}{2! \times 2!} = 30$

Count the numbers with two 4s and two 7s.

Number with two 4s and two 7s $= 4 \times 30 = 120$

Count the combinations with three 4s and two 7s.

There is 1 combination with three 4s and two 7s.

Count the permutations.

Permutations of three 4s and two 7s $= \dfrac{5!}{2! \times 3!} = 10$

Count the numbers with three 4s and two 7s.

Number with three 4s and two 7s $= 1 \times 10 = 10$

Use the addition principle.

Total number $= 720 + 600 + 600 + 200 + 120 + 10 = 2250$

State the answer.

There are 2250 possible 5-digit numbers.

Sometimes the inclusion-exclusion principle, or a variation of it, can be used to solve a problem more efficiently than by using only using combinations or permutations.

Example 15

Peter, Freda and Elena go to buy tickets for a concert. They travel to the box office separately, so their positions in the queue are essentially random. There are 10 people in the queue.

- a In how many different ways could Peter and Freda be together in the queue?
- b In how many ways could all three friends be together in the queue?
- c In how many ways could they all be separated?

Solution

a Consider Peter and Freda as one unit.

The number of permutations of 9 items = 9!

In how many orders can Peter and Freda be?

The number of permutations of 2 items = 2!

Use the multiplication principle.

Ways Peter and Freda are together $= 9! \times 2!$
$= 725\,760$

Write the answer.

There are 725 760 ways Peter and Freda could be together in the queue.

b Consider the three friends as a group.

The number of permutations of 8 items = 8!

In how many orders can the three be?

The number of permutations of 3 items = 3!

Use the multiplication principle.

Ways the three are together $= 8! \times 3!$
$= 241\,920$

Write the answer.

There are 241 920 ways that all three friends could be together in the queue.

c Find the number of possible queues.

Number of queues $= 10!$
$= 3\,628\,800$

State the ways that they can be together.

All three can be together or two can be together.

State the overlap.

The number of any two together includes the three together in some orders.

Peter and Freda together includes EFP, EPF, FPE and PFE, but not FEP or PEF.

In this case, two together includes three together, but only in four orders, not all six.

State the actual intersection, considering the three as one unit.

The number in the intersection of Peter and Freda together with all three together $= 4 \times 8!$
$= 161\,280$

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Two together can be Peter and Freda, Peter and Elena or Elena and Freda.	Each of the three ways that two friends can be together has the same number.
Use the inclusion exclusion principle.	Number of ways some of the friends can be together $= 3 \times 725\ 760 - 3 \times 161\ 280 + 241\ 920$
Calculate the number.	$= 1\ 935\ 360$
Find the number of ways they are separated.	Number of ways all separate $= 3\ 628\ 800 - 1\ 935\ 360$ $= 1\ 693\ 440$
State the answer.	There are 1 693 440 ways that the 3 friends can be separated in the queue of 10 people.

EXERCISE 5.06 General use of counting methods

Reasoning and communication

1 Example 13 How many different 4-letter 'words' can be made from the letters of SELECTED if, in making a word, each of the 8 letters can be used only once in each 'word'?

2 Example 14 How many numbers of each of the following lengths can be made from the digits 1, 2, 2, 3, 4, 5, 5 if each of the 7 digits can be used only once in each number?

a 2 digits　　b 3 digits　　c 4 digits　　d 5 digits

3 Example 15 Seven people, including three friends, are seated in row at random. In how many ways can each of the three friends be seated apart from each other?

4 How many different 4-letter 'words' can be made from the letters of ENTERPRISES if, in making a word, each of the 11 letters can be used only once in each 'word'?

5 How many different 5-letter 'words' can be made from the letters of SYNTHESIS if, in making a word, each of the 9 letters can be used only once in each 'word'?

6 Four friends at an amusement park are in the line to get on a 'terror ride'. They are all in the first 20 in the queue, so they will all get on the next ride.

a In how many ways could John and Mary be together?

b In how many ways could John, Mary and Massoud be together?

c In how many ways could all four be together?

d In how many ways could they all be separated?

7 A restaurant has round tables that seat 8 people and square tables that seat 4 people. In how many ways could a party of 12 people be seated using one 4-person and one 8-person table?

5.07 GENERAL APPLICATIONS TO PROBABILITY

Probability problems are mostly calculated by counting the number of ways that things might happen. When solving them, you need to work out the kind of counting involved: addition, multiplication, pigeonhole or inclusion-exclusion principles, permutations or combinations or multiple methods.

Example 16

In melding card games like Canasta and Gin Rummy, players attempt to make three or four of a kind or to make runs of three or more cards of the same suit. What is the probability that the first three cards a player receives is a run of three cards of the same suit (with Ace high)?

Solution

State the melds in one suit. — The spade melds are 2-3-4, 3-4-5, …., 10-J-Q, J-Q-K, Q-K-A

Count them. — There are 11 possible spade melds of 3 cards.

There are 4 suits. — Number of possible melds altogether $= 4 \times 11$
$= 44$

Count the possible 3-card hands. — Number of possible 3-card hands $= {}^{52}C_3$
$= 22\ 100$

Work out the probability. — Probability of a 3-card run $= \dfrac{44}{22\ 100}$

$= \dfrac{11}{5525}$

$\approx 0.001\,99$

Round and write the answer. — There is about a 0.2% probability of a 3-card run in the first three cards.

When doing multiple calculations, it is often easier to evaluate everything at the end.

Example 17

Four marbles are taken at random from a bag containing 5 red, 3 green and 4 yellow marbles. What is the probability that two marbles are red and the other two are yellow?

Solution

The order is not important. — The number of ways of taking 4 marbles $= {}^{12}C_4$

You want 2 red and 2 yellow. — Use the multiplication principle, so $n(\text{2R and 2Y}) = {}^5C_2 \times {}^4C_2$

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Write the probability.	$P(\text{2R and 2Y}) = \dfrac{{}^5C_2 \times {}^4C_2}{{}^{12}C_4}$
	$= {}^5C_2 \times {}^4C_2 \div {}^{12}C_4$
Substitute formulas.	$= \dfrac{5\times 4}{1\times 2} \times \dfrac{4\times 3}{1\times 2} \div \dfrac{12\times 11\times 10\times 9}{1\times 2\times 3\times 4}$
Multiply by the reciprocal.	$= \dfrac{5\times 4}{1\times 2} \times \dfrac{4\times 3}{1\times 2} \times \dfrac{1\times 2\times 3\times 4}{12\times 11\times 10\times 9}$
Cancel and evaluate.	$= \dfrac{4}{33}$
	≈ 0.12
Write the answer.	The probability of getting 2 red and 2 yellow marbles is $\frac{4}{33}$, or about 12%.

EXERCISE 5.07 General applications to probability

Reasoning and communication

1 Example 16 In the card game pontoon, the highest hand (a pontoon) consists of an ace and a court card (J, Q, K) or 10. Each player is dealt two cards to start with. What is the probability of getting a pontoon?

2 Example 17 A box of spare parts has 18 pistons with a tolerance of 0.05 mm in the diameter. There are 8 oversized pistons and 10 undersized pistons in the box. Four pistons are taken from the box at random to rebuild an engine. What is the probability that they are:

a all undersized?

b all oversized?

c half undersized and half oversized?

3 A full hand in poker has a pair and three of a kind, such as 2 fives and 3 kings. What is the probability of being dealt a full house from a normal pack?

4 A flush in poker has 5 cards of the same suit. What is the probability of being dealt a flush from each of the following packs?

a A normal 52 card pack

b A euchre pack (7s up) of 32 cards

c A 6-handed 500 pack that includes 11s, 12s and red 13s in addition to the normal 52 cards.

5 In a busy office, three letters are prepared with envelopes addressed, ready for posting to three different people. Unfortunately, the letters and envelopes are dropped on the floor and mixed up. The person who is asked to place the letters in the envelopes is very careless and doesn't pay attention to the address on the envelopes when the letters are placed in them. What is the probability that no one gets the right letter?

6 25 students were selected at random from a group of Year 11 students who received a grade of A, B or C on a test. There were 30 who got As, 50 who got Bs and 80 who got Cs.

a What is the probability that exactly 15 of the students selected had the same grade?

b What is the probability that at least 10 students had the same grade?

c What is the smallest number n such that the probability of 10 students getting the same grade when n are selected is 1?

7 A band has four guitarists, each of whom has their own guitar. The guitars all look similar and it is common for the guitarists to pick up someone else's guitar by mistake.

a What is the probability of exactly three picking up the wrong guitar?

b What is the probability of at least three picking up the wrong guitar?

c What is the probability that no one picks up the right guitar?

8 Max's mother is colour blind. In his wardrobe he has 3 green shirts, 2 red shirts, 1 blue shirt, 2 black pairs of pants and 3 cream pairs of pants. His mother picks some clothes for him to wear while he is in the shower, but Max is particular about what he wears and will only wear outfits where the shirt and pants colours 'go with' each other. In Max's view, black pants only go with blue or red shirts, and cream pants only go with red or green shirts. What is the probability that Max's mother has picked an acceptable outfit?

9 Ten friends were all on holidays in different parts of the world. They all sent postcards to five different friends from the ten, essentially choosing the 5 at random. What is the probability that at least two people sent cards to each other?

10 To enter a building, workers must enter a code into a touchpad at the entrance. The code has 6 digits. Helen remembers 5 of the digits for her code, but can't remember the order. She knows that the digits are all different. What is the probability of her getting the correct code in the three goes she is allowed?

11 On the first day of the school year, a teacher assigns desks in his classroom to students at random as they enter the classroom. The desks are set out together in groups of 3. There are 24 seats and 24 students in one class.

a What is the probability that three friends are seated together?

b What is the probability that they are seated apart (at different sets of 3 desks)?

12 Some students arrive at the basketball court and decide to make up two teams to play a practice game. Three of the players are much better than the others and a team where all three are on the same side is too strong. The teams are chosen at random using 'eeny, meeny, miny, moe'. The last person must be the umpire. What is the probability of one team being too strong if there are

a 7 students? b 9 students? c 11 students? d 15 students?

(*Note:* The odd one out must umpire!)

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CHAPTER SUMMARY

5 PERMUTATIONS AND COMBINATIONS

- **Combinatorics** is the mathematics of systematically counting finite collections
- A **combination** is an unordered selection. A combination of a set of symbols is a selection of none, some or all of the symbols.
- The number of combinations of r objects from n distinct objects is given by

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

- The **addition principle** states that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.
- The **multiplication principle** states that, for a choice made in 2 stages, if there are a ways for one part and b ways for the second part, then there are $a \times b$ choices altogether. If a choice is made in n stages, each of which have a_i possibilities, then there are $a_1a_2a_3 \ldots a_n$ possible choices altogether.
- Pascal's triangle is an arrangement of staggered numbers in rows such that each row begins and ends with 1 and the other numbers are the sums of the two numbers immediately diagonally above them.

						1							Row 0
					1		1						Row 1
				1		2		1					Row 2
			1		3		3		1				Row 3
		1		4		6		4		1			Row 4
	1		5		10		10		5		1		Row 5
1		6		15		20		15		6		1	Row 6

- The numbers in Pascal's triangle are the combinatorial numbers given by ^nC_r and are also the binomial coefficients, the coefficients in the expansion of $(x+y)^n$.
- The **inclusion-exclusion principle** states that the number of elements in a union of sets is given by:

$|A \cup B| = |A| + |B| - |A \cap B|$ for 2 sets

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$
$- |A \cap C| - |B \cap C| + |A \cap B \cap C|$
for 3 sets

The pattern of inclusions and exclusions extends to any number of sets.

- The number of ways of arranging n **objects in an ordered list** is given by
$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
- A **permutation** is an ordered arrangement. A permutation of a set of symbols is a selection of none, some or all of the symbols in a particular order.
- The number of permutations of r objects from n distinct objects is written as ^nP_r and is given by

$$^nP_r = \frac{n!}{(n-r)!}$$
$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

- **Circular permutations** are ordered arrangements in a circle. The number of circular permutations of n different objects or symbols is $(n-1)!$.
- The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$.

If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \ldots \le n$.

5

CHAPTER REVIEW

COUNTING METHODS AND COMBINATIONS

Multiple choice

1 Example 1 The number of combinations of 4 symbols from 8 different symbols is:
A 24 B 64 C 70 D 420 E 1680

2 Example 2 The number of committees of 3 that can be chosen from 10 people is:
A 30 B 120 C 270 D 720 E 810

3 Example 8 The coefficient of x^3yz in the expansion of $(x+y+z)^5$ is:
A 10 B 12 C 15 D 20 E 50

4 Example 9 The number of natural numbers up to 200 divisible by 3 or 5 is:
A 92 B 93 C 106 D 107 E 108

5 Example 16 The probability that the first 4 cards dealt from a normal pack make two (different) pairs like 4 K K 4 is:
A about 0.000 044 3 B about 0.003 53 C about 0.004 61
D about 0.006 41 E about 0.010 37

Short answer

6 Example 4 How many different committees of four people including Mohammed and Julius could be chosen from a group of fifteen people?

7 Example 5 A netball team of 7 players is to be chosen from a squad of twelve players. How many different teams can be chosen that include Kylie or Simone, but not both?

8 Example 6 A class of 26 students with 14 girls and 12 boys has three representatives on the student council.
a In how many ways can the three be chosen with no restrictions?
b In how many ways can they be chosen if at least one person of each sex must be included?

9 Example 10 How many positive whole numbers under 1000 are divisible by 5, 7 or 9?

10 Example 11 35 of the cows in a dairy herd have horns and 9 of these are a uniform colour. Altogether, there are 72 cows in the herd that have either horns or a uniform colour. There are 220 cows altogether in the herd. What is the probability that a cow chosen at random for testing is of uniform colour?

11 Example 12 In a group of 200 people, 70 have fair skin and the rest have olive or dark complexions. 80 have light-coloured hair (fair or red) and the rest have dark hair. There are 90 with light-coloured eyes (blue, grey, green or hazel) and the rest have dark-coloured eyes. Of those with fair skin 30 have light-coloured hair and 30 have light-coloured eyes. 50 have light-coloured eyes and hair. Of this 50, 20 also have fair skin. How many people in the group have at least one of fair skin or light-coloured eyes or hair?

9780170250276

12 Example 13 How many different 4-letter 'words' can be made from the letters of ENVELOPE if, in making a word, each of the 8 letters can be used only once in each 'word'?

13 Example 17 A box of printer cartridges has eight new cartridges and 10 remanufactured cartridges. What is the probability that when six are taken out at random, equal numbers of new and remanufactured cartridges are selected?

Application

14 Prove that ${}^{n}C_{r} = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$.

15 Prove that if the first number after the 1 in a row of Pascal's triangle is prime, then it divides all the other numbers, apart from the 1s at either end.

16 In the Australian version of the card game Euchre, the cards 7, 8, 9, 10, J, Q, K and A of each suit are used, making 32 cards altogether. One suit is chosen as trumps, and the Jack of the same colour is included in the trump suit as the second highest card (the left Bower). How many hands of 5 cards are possible that include exactly

a 4 trumps b 3 trumps c no trumps?

Practice quiz

17 Three couples go to a concert together and they all sit together. In how many ways can everyone be seated so that no couple is together?